Criteria for unbiased estimation: Applications to noise-agnostic sensing and channel learning

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arXiv: 2503.17362

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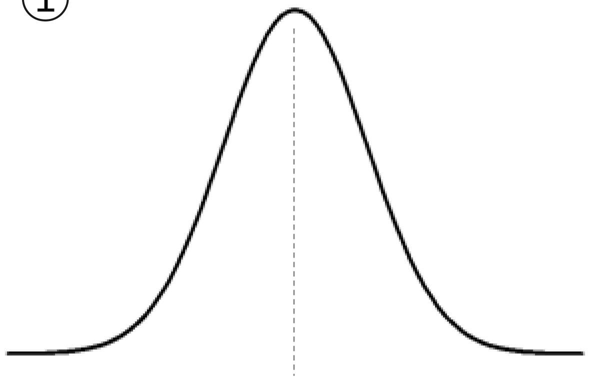
Pop Quiz

Can you accurately find what are θ_1 and θ_2?

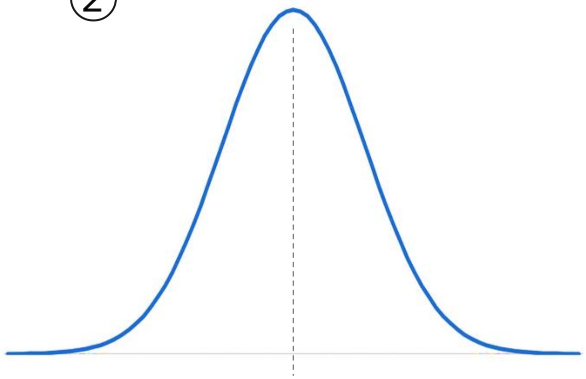
①



$\mu = \theta_1$

$\sigma = 1$

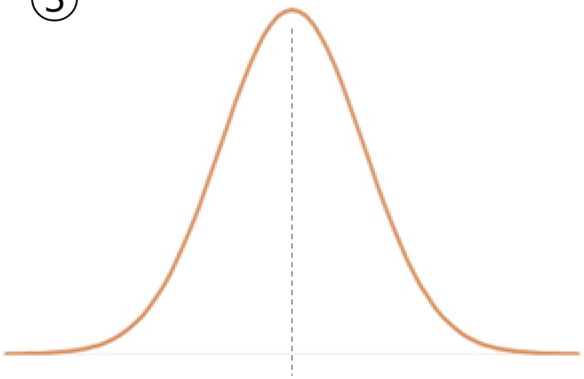
②



$\mu = \theta_1 + \theta_2$

$\sigma = 1$

③



$\mu = \theta_1$

$\sigma = \theta_2$

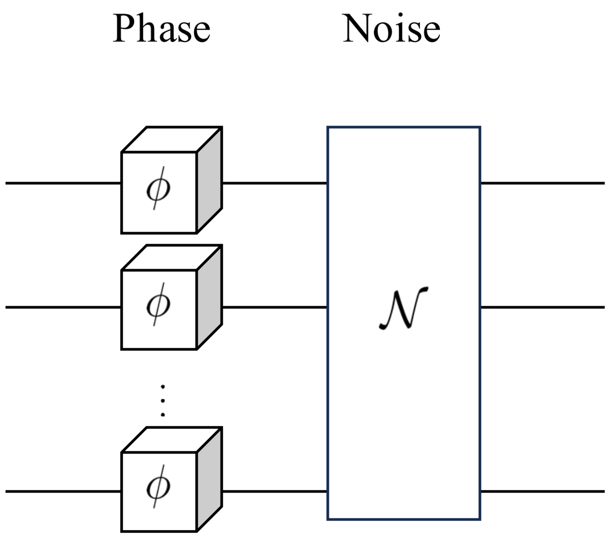
Is there any general criteria for achieving unbiased estimation?

Multi-parameter estimations

Quantum systems are influenced by multiple parameters

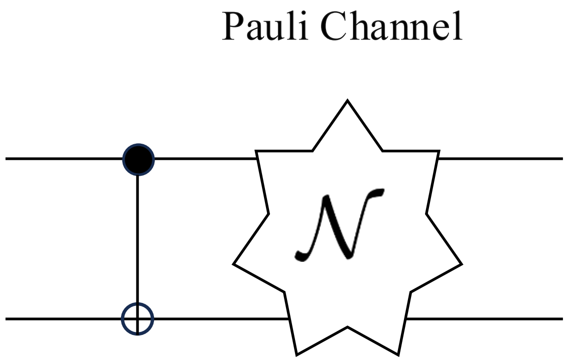
- Phase estimation

Phys. Rev. A **89,** 023845 (2014).
Nat. Commun. **5,** 3532 (2014).



- Channel learning

Nat. Commun. **14,** 52 (2023).

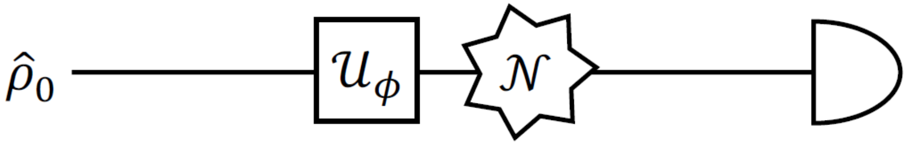


In this work

We establish necessary and sufficient conditions for unbiased estimation

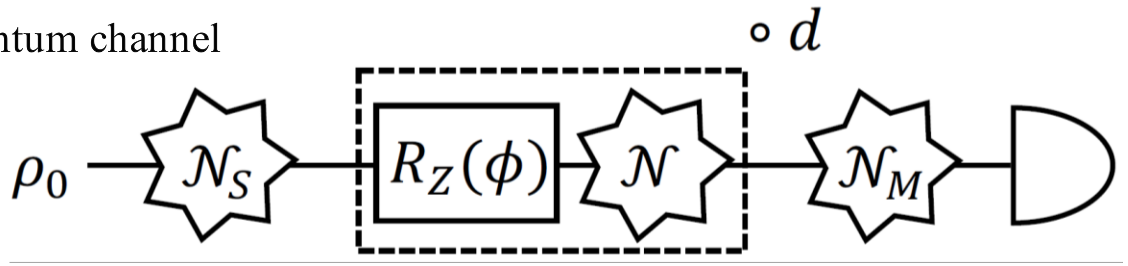
State estimation: parameters are encoded in a quantum state

- Lemma 1: based on quantum Fisher information matrix
- Theorem 1: based on the derivatives of the encoded state
- Application: noise agnostic-sensing

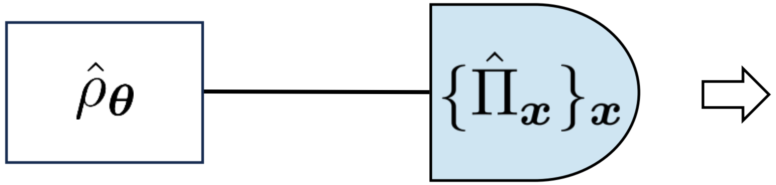


Channel estimation: estimating parameters characterizing quantum channel

- Corollary 1: based on the derivatives of the quantum channel
- Application: learnability of quantum channel



Multi-parameter estimation and Unbiased estimator



$\boldsymbol{\theta} := \{\theta_1, \theta_2, \cdots, \theta_M\}$

POVM

$\tilde{\boldsymbol{\theta}} := \{\tilde{\theta}_1(\boldsymbol{x}), \tilde{\theta}_2(\boldsymbol{x}), \cdots, \tilde{\theta}_M(\boldsymbol{x})\}$

(estimated values of $\boldsymbol{\theta}$)

Unbiased estimator

$$\langle \tilde{\theta}_i(\boldsymbol{x}) \rangle := \int \tilde{\theta}_i(\boldsymbol{x}) \, p(\boldsymbol{x}) d\boldsymbol{x} = \theta_i$$

Can we always perform unbiased estimation for every parameters?

Quantum Cramér–Rao bound (QCRB)

When quantum Fisher information matrix is invertible, unbiased estimations are possible!

Quantum Cramér–Rao lower bound J. Phys. A: Math. Theor. **53,** 453001 (2020).

$$\boldsymbol{w}_1^{\mathrm{T}}\mathbf{C}\boldsymbol{w}_1 = \left\langle \left(\tilde{\theta}_1 - \langle\tilde{\theta}_1\rangle\right)^2 \right\rangle \geq \boldsymbol{w}_1^{\mathrm{T}}\mathbf{J}^{-1}\boldsymbol{w}_1 \qquad \boldsymbol{w}_1 = (1, 0, \cdots, 0)^{\mathrm{T}}$$

$\mathbf{C}$ is $M \times M$ covariance matrix with its elements $[\mathbf{C}]_{ij} = \langle(\tilde{\theta}_i - \langle\tilde{\theta}_i\rangle)(\tilde{\theta}_j - \langle\tilde{\theta}_j\rangle)\rangle$

$\mathbf{J}$ is $M \times M$ quantum Fisher information matrix with its elements $[\mathbf{J}]_{ij} = \mathrm{Tr}\left[\hat{\rho}_{\boldsymbol{\theta}}\{\hat{L}_i, \hat{L}_j\}\right]$ $\left(\frac{\partial\hat{\rho}_{\boldsymbol{\theta}}}{\partial\theta_i} = \frac{1}{2}\{\hat{L}_i, \hat{\rho}_{\boldsymbol{\theta}}\}\right)$

- There always exist an **unbiased estimator of $\boldsymbol{\theta}_1$** and **measurement** that saturate the **equality**
- This can be generalized to arbitrary parameter $\boldsymbol{\phi} = \boldsymbol{w}^T\boldsymbol{\theta}$ for any given $\boldsymbol{w} \in \mathbb{R}^M$

1. Unbiased estimation for any given parameter $\boldsymbol{\phi} = \boldsymbol{w}^T\boldsymbol{\theta}$ is possible
2. Provides ultimate achievable estimation error

(When QFIM is invertible!)

Simple examples for non-invertible QFIM

When QFIM is not invertible,

1. Unbiased estimation of $\boldsymbol{\phi} = \boldsymbol{w}^T\boldsymbol{\theta}$ cannot be guaranteed
2. Achievable lower bound (QCRB) cannot be defined

Example 1. $e^{-i\theta_1\hat{Z}}e^{-i\theta_2\hat{X}}\left|+\right\rangle = e^{-i\theta_1\hat{Z}}\left|+\right\rangle$ $\mathbf{J} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$

parameters	θ_1	θ_2	$\theta_1 + \theta_2$	$\theta_1 - \theta_2$
unbiasedness	O	X	X	X

Example 2. $e^{-i(\theta_1+\theta_2)\hat{Z}}\left|+\right\rangle$ $\mathbf{J} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ $\mathbf{J}_{\mathrm{SVD}} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$

parameters	θ_1	θ_2	$\theta_1 + \theta_2$	$\theta_1 - \theta_2$
unbiasedness	X	X	O	X

⇨ Parameters related to zero part of QFIM are not unbiasedly estimatable

Lemma 1.

[Related works: IEE Signal Process. Lett. **16,** 453, IEE Trans. Signal Process. **60,** 5532, IEEE Signal Process. Lett. **5,** 177, IEEE Trans. Signal Process. **49,** 87, Int. J. Quantum Inf. **19,** 2140004, Phys. Rev. A **111,** 012414, arXiv:2412.01117]

Lemma 1. Unbiased estimation of $\phi = \boldsymbol{w}^{\mathrm{T}}\boldsymbol{\theta}$ can be performed if and only if $\boldsymbol{w} \in \operatorname{supp}(\mathbf{J})$. If the condition is satisfied, the achievable lower bound of the estimation error is given by generalized QCRB:

$$\Delta^2\phi = \boldsymbol{w}^{\mathrm{T}}\mathbf{C}\boldsymbol{w} \geq \boldsymbol{w}^{\mathrm{T}}\mathbf{J}^{+}\boldsymbol{w} \tag{1}$$

- supp($\boldsymbol{J}$): subspace spanned by non-zero eigenvectors of $\boldsymbol{J}$
- $\boldsymbol{J}^{+}$: Moore-Penrose pseudoinverse. The pseudo inverse on the support of J is equivalent to the inverse defined on the support of J.

- Necessary and sufficient condition for unbiased estimation
- Generalized QCRB (Eq. (1)) which provides the lower bound and can be defined even when QFIM is not invertible

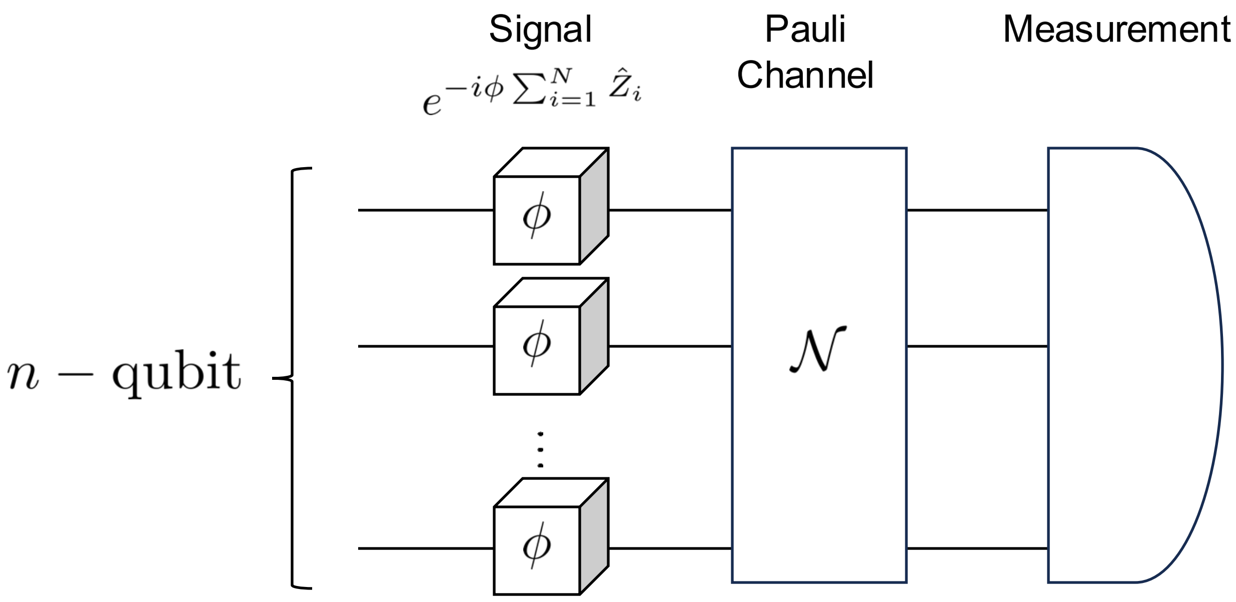
Theorem 1.

Theorem 1. Unbiasd estimation of θ_1 can be performed if and only if

$$\frac{\partial \hat{\rho}_{\boldsymbol{\theta}}}{\partial \theta_1} \neq \sum_{i=2}^{M} c_i \frac{\partial \hat{\rho}_{\boldsymbol{\theta}}}{\partial \theta_i}, \ \forall c_i \in \mathbb{C}. \tag{1}$$

- If Eq. (1) is violated, there exists at least two different parameter set which results in the same encoded state $\hat{\rho}_{\boldsymbol{\theta}}$ → Cannot perform unbiased estimation
- Equivalent to Lemma 1, easier to detect

Application: Noise agnostic sensing



Unknown Pauli channel $\mathcal{N}$

- Pauli eigenvalues

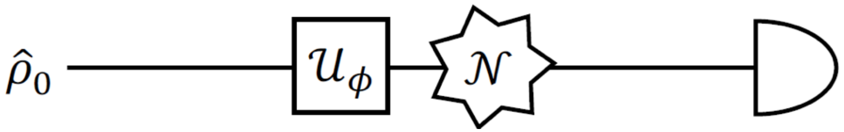
$$\mathcal{N}(\hat{P}_{\boldsymbol{a}}) = \lambda_a \hat{P}_{\boldsymbol{a}}$$

- Assume that all the Pauli eigenvalues are unknown

Can perform unbiased estimation of ϕ?

Application: Noise agnostic sensing

(a) Naive sensing protocol

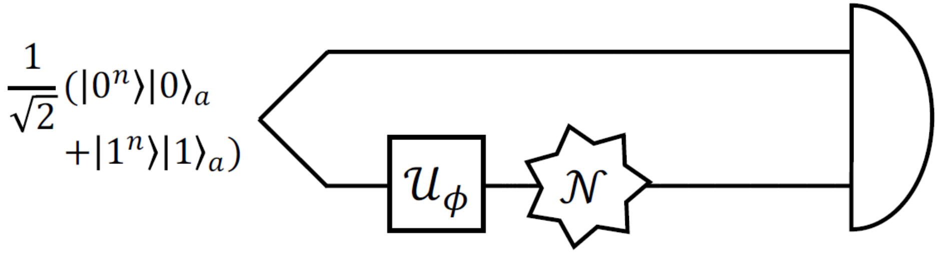


$$\hat{\rho}_0 = \sum_{\boldsymbol{a}} \hat{P}_{\boldsymbol{a}} \longrightarrow \sum_{\boldsymbol{a}} u_{\boldsymbol{a}}(\phi)\hat{P}_{\boldsymbol{a}} \longrightarrow \hat{\rho} = \sum_{\boldsymbol{a}} \lambda_{\boldsymbol{a}} u_{\boldsymbol{a}}(\phi)\hat{P}_{\boldsymbol{a}}$$

$$\Rightarrow \partial_\phi \hat{\rho} = \sum_{\boldsymbol{a}} \frac{\partial_\phi u_{\boldsymbol{a}}(\phi)}{u_{\boldsymbol{a}}(\phi)} \lambda_{\boldsymbol{a}} \partial_{\lambda_{\boldsymbol{a}}} \hat{\rho}$$

- Unbiased estimation: X

(b) Noise-agnostic sensing using entanglement



- Unbiased estimation: O
- The ultimate achievable lower bound is (according to generalized QCRB)

$$\delta^2\phi \geq \frac{1}{n^2 \sum_{\boldsymbol{x}\in\{0,1\}^n} p_{\boldsymbol{x}} \lambda_{\boldsymbol{x}}^2}$$

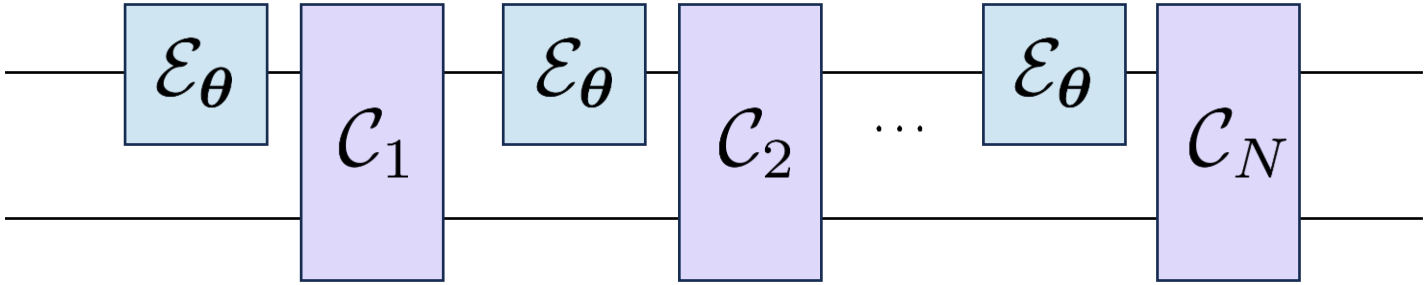
Noiseless ancilla enables unbiased estimation

Corollary 1.

Corollary 1. For a given quantum channel $\mathcal{E}_{\boldsymbol{\theta}}$, unbiasd estimation of θ_1 can be performed if and only if,

$$\frac{\partial \mathcal{E}_{\boldsymbol{\theta}}}{\partial \theta_1} \neq \sum_{i \neq 1} c_i \frac{\partial \mathcal{E}_{\boldsymbol{\theta}}}{\partial \theta_i}, \ \forall c_i \in \mathbb{C}. \tag{1}$$

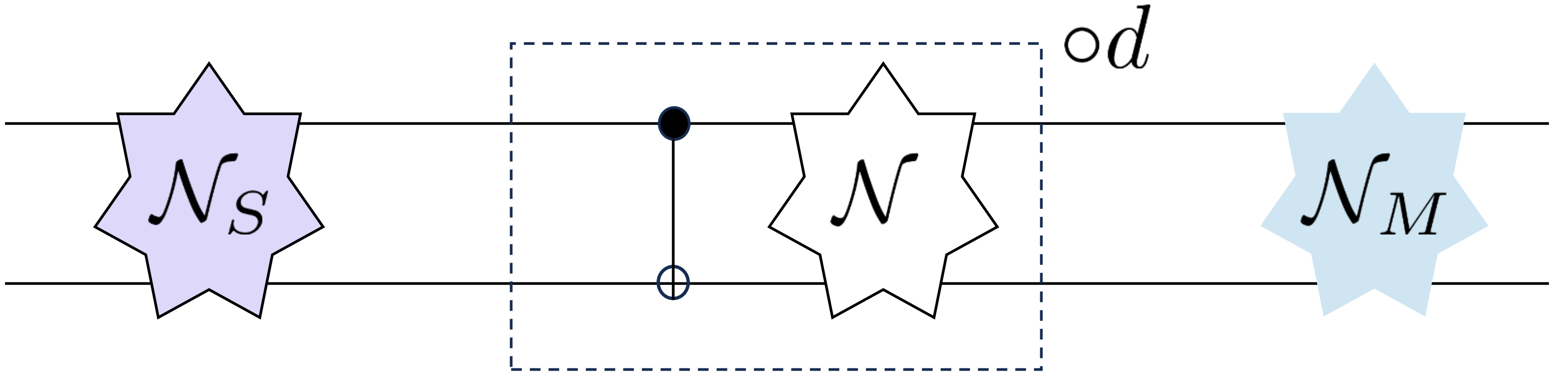
- If Eq. (1) is violated, even the sequential scheme cannot perform unbiased estimation



Learnability of Pauli channel

Nat. Commun. **14,** 52 (2023).

Pauli channel on CNOT gate



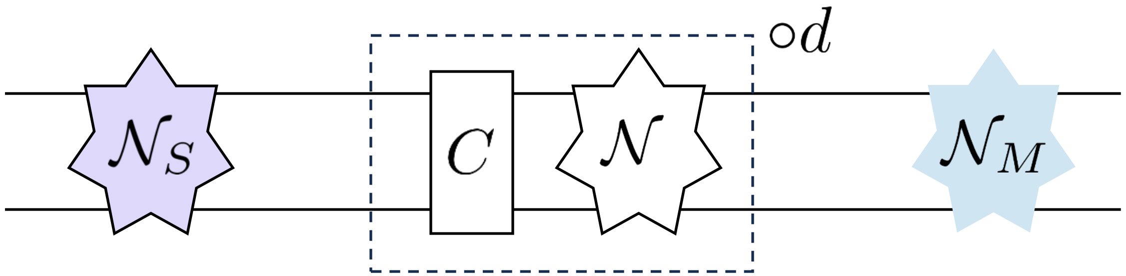
- Can we learn all the Pauli eigenvalues of $\mathcal{N}$? ⇨ Yes
- Can we learn all the Pauli eigenvalues of $\mathcal{N}$ in the presence of SPAM error? ⇨ No

Because there is gauge degree of freedom $\{\mathcal{N}, \mathcal{N}_S, \mathcal{N}_M\} = \{\mathcal{N}', \mathcal{N}'_S, \mathcal{N}'_M\}$

Learnability of Pauli channel

Ref. Nat. Commun. **14,** 52

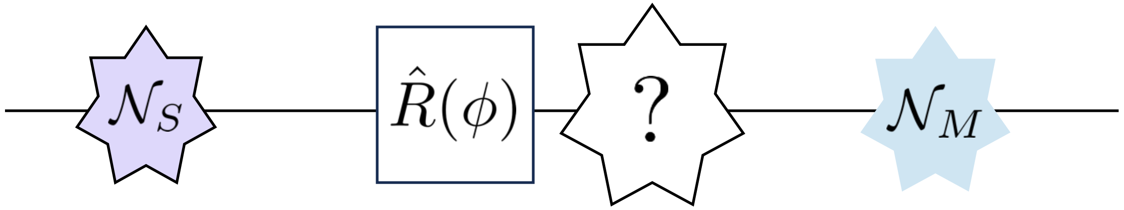
Classified learnable and un-learnable Pauli eigenvalues for Pauli channel acting on two-qubit Clifford gate



- Proof depends on the structure of the Clifford gates and Pauli channel

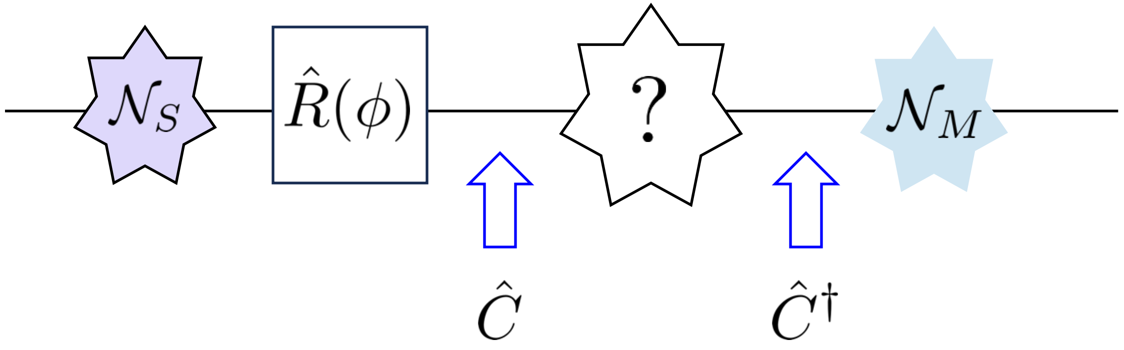
⇨ Can we generalize this to non-Clifford gate and non-Pauli noise?

Phase rotation gate + Symmetric Clifford twirling

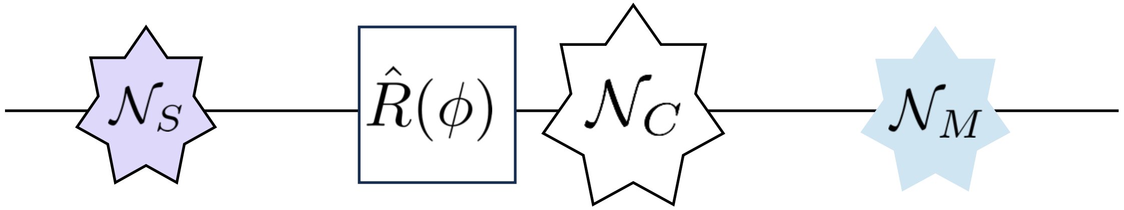


$$\hat{R}(\phi) = e^{-i\hat{Z}\phi/2}$$

- Assume SPAM noises are Pauli noises: $\mathcal{N}_S,\ \mathcal{N}_M$



- Applying Random Clifford gate s.t. $[\hat{C}, \hat{Z}] = 0$

$=$ 

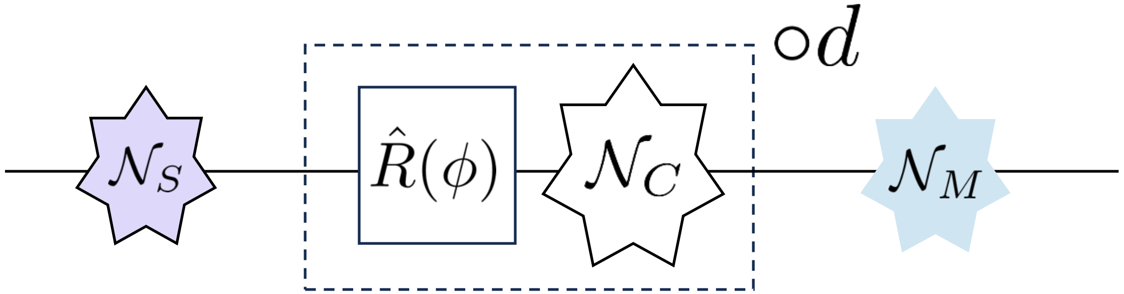
$$\mathcal{N}_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 \cos\theta & \lambda_1 \sin\theta & 0 \\ 0 & -\lambda_1 \sin\theta & \lambda_1 \cos\theta & 0 \\ \alpha & 0 & 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \hat{I} \\ \hat{X} \\ \hat{Y} \\ \hat{Z} \end{pmatrix}$$

- Arbitrary noise → [depolarizing + dephasing + amplitude damping + coherent noise]

⇨ Can we learn the relevant parameters $\lambda_1, \lambda_2, \theta, \alpha$?

arXiv:2405.07720 (2024).

Noise learnability through Cycle benchmarking



$$\mathcal{N}_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 \cos\theta & \lambda_1 \sin\theta & 0 \\ 0 & -\lambda_1 \sin\theta & \lambda_1 \cos\theta & 0 \\ \alpha & 0 & 0 & \lambda_2 \end{pmatrix}$$

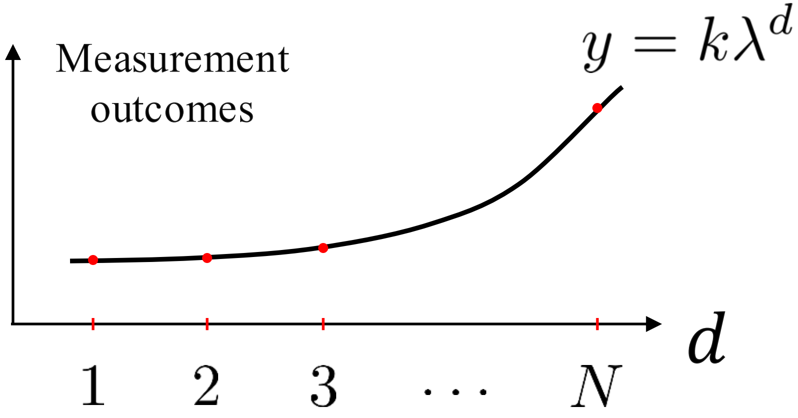
Cycle benchmarking Nat. Commun. 10, 5347 (2019).

- Cycle benchmarking identifies noise parameters by repeatedly applying a gate sequence and analyzing the resulting outputs to extrapolate the parameters

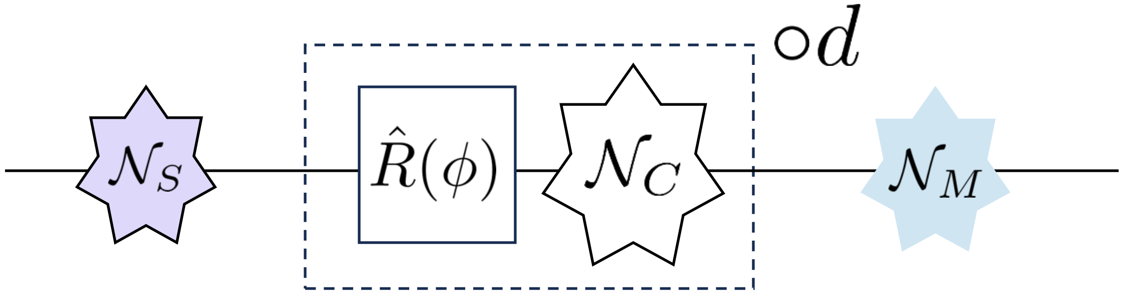
- Quantum channel

$$\mathcal{E}_{\text{cycle}} : \hat{\rho} \otimes \hat{\rho}_c \mapsto \sum_d \mathcal{N}_d(\hat{\rho}) \otimes |d\rangle\langle d|\, \hat{\rho}_c\, |d\rangle\langle d|$$

$$\mathcal{N}_d = \mathcal{N}_M \circ (\mathcal{N} \circ \mathcal{U})^{\circ d} \circ \mathcal{N}_S$$



Noise learnability through Cycle benchmarking



$$\mathcal{N}_C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 \cos\theta & \lambda_1 \sin\theta & 0 \\ 0 & -\lambda_1 \sin\theta & \lambda_1 \cos\theta & 0 \\ \alpha & 0 & 0 & \lambda_2 \end{pmatrix}$$

Learnability of the parameters of $\mathcal{N}_C(\lambda_1, \lambda_2, \theta, \alpha)$

- α is not learnable = unbiased estimation of α is impossible

Violation of Corollary 1: $\alpha \partial_\alpha \mathcal{E}_{\text{cycle}} = \lambda_{3M} \partial_{\lambda_{3M}} \mathcal{E}_{\text{cycle}} - \lambda_{3S} \partial_{\lambda_{3S}} \mathcal{E}_{\text{cycle}}$

(λ_{3S}, λ_{3M} : Noise parameters of SPAM noise)

- $\lambda_1, \lambda_2, \phi$ are learnable = we can perform unbiased estimation even though we do not know SPAM noise

Conclusion

We derive necessary and sufficient conditions for the unbiased estimation for state and channel estimation

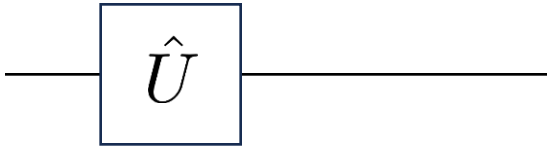
Quantum metrology

- Inspecting feasibility of unbiased estimation.

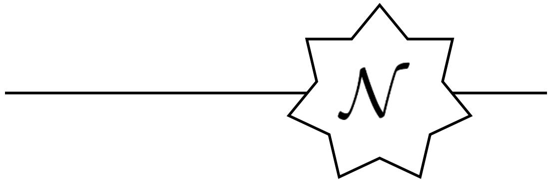
Learning theory

- Inspecting learnability of channel parameters

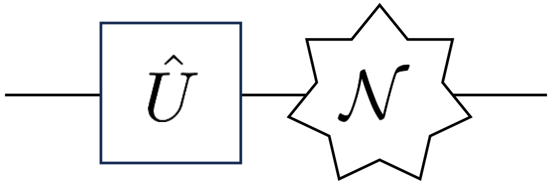
Hamiltonian learning arXiv: 2502. 11900



Pauli channel learning Phys. Rev. Lett. **132,** 180805



Unitary + Pauli channel learning



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